INFLUENCE OF STREAM VORTICITY ON FRICTION AND HEAT EXCHANGE IN APPLICATION TO THE CASE OF A SUPERSONIC JET IMPINGING ON AN OBSTACLE

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The flow in the neighborhood of the stagnation point of a stationary plane stream of viscous incompressible fluid with harmonic components superposed on the velocity components is investigated. Such a flow corresponds to a stream containing periodic eddies perpendicular to the flow plane. The results of the investigation can be applied to estimate thermal fluxes.

Let us consider a stationary plane stream of viscous incompressible fluid in which stationary periodic eddies oriented perpendicularly to the flow plane $\xi 0 \eta$ (the ζ axis) are introduced. The flow is bounded by the $\eta = 0$ plane with the stagnation point $\xi = \eta = 0$.

Let us utilize the following differential equations:

the equation of vortex transport in a projection on the ζ axis

$$v_x \frac{\partial \Omega_z}{\partial x} + v_y \frac{\partial \Omega_z}{\partial y} = \frac{\partial^2 \Omega_z}{\partial x^2} + \frac{\partial^2 \Omega_z}{\partial y^2}, \tag{1}$$

where

$$\Omega_z = \frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x}; \quad v_x = \frac{\partial \psi}{\partial y}; \quad v_y = -\frac{\partial \psi}{\partial x};$$

the energy equation (without taking account of dissipation)

$$v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} = \frac{1}{\Pr}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right). \tag{2}$$

The following dimensionless variables are utilized here

$$\begin{gathered} x = \sqrt{\frac{\beta}{\nu}}\,\xi; \quad y = \sqrt{\frac{\beta}{\nu}}\,\eta; \quad v_x = \frac{v_\xi}{\sqrt{\beta\nu}}; \\ v_y = \frac{v_\eta}{\sqrt{\beta\nu}}; \quad \Omega_z = \frac{\Omega_\zeta}{\beta}; \quad T(x, y) = \frac{T_w - T(\xi, \eta)}{T_w - T_\infty}. \end{gathered} \tag{3}$$

It is known that under the boundary conditions

$$\begin{gathered} v_x(x, 0) = v_y(x, 0) = T(x, 0) = 0; \\ v_x(x, \infty) = x; \quad \frac{\partial v_y(x, \infty)}{\partial y} = -1; \quad T(x, \infty) = 1 \end{gathered} \tag{4}$$

the velocity components for a stationary viscous fluid flow in the neighborhood of the stagnation point and in the absence of initial stream pulsations are [3]

$$v_x = xf'(y); \quad v_y = -f(y),$$

where f(y) is found from the solution

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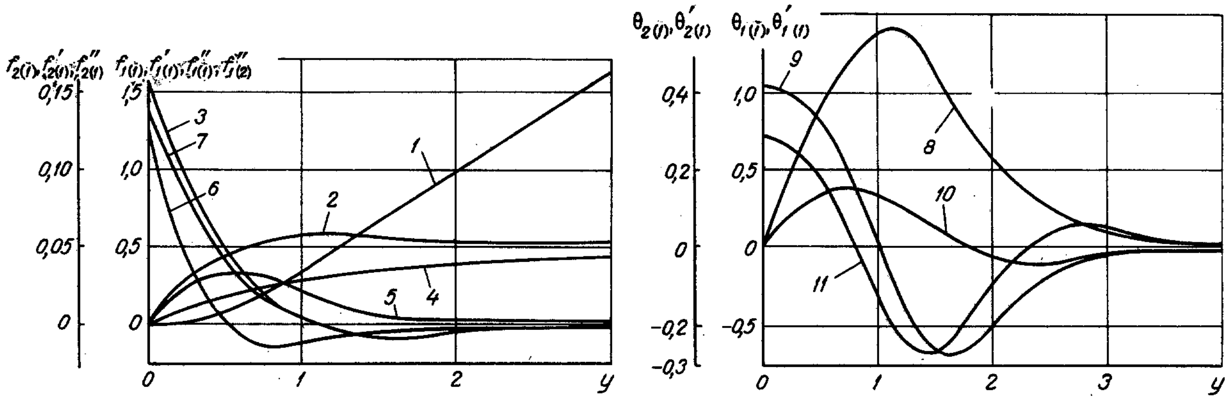

Fig. 1. Harmonic velocity components and temperatures in the expansions (7), (8): 1) $f_{1(1)}$; 2) $f'_{1(1)}$; 3) $f''_{1(1)}$; 4) $f_{2(1)}$; 5) $f'_{2(1)}$; 6) $f''_{2(1)}$; 7) $f''_{1(2)}$; 8) $\Theta_{1(1)}$; 9) $\Theta'_{1(1)}$; 10) $\Theta_{2(1)}$; 11) $\Theta'_{2(1)}$.

$$f'^2 - ff'' = 1 + f''' \quad \text{when} \quad f(0) = f'(0) = 0; \quad f'(\infty) = 1$$

(here and henceforth the prime will denote differentiation with respect to y).

The presence of periodically distributed vortices along the x axis assumes the appearance of harmonic components for the velocity components at infinity, hence the boundary conditions (4) can be written as:

$$\begin{gathered} y = 0 \quad v_x = 0; \quad v_y = 0, \\ y = \infty \quad \frac{\partial v_y}{\partial y} = -1 - \sum_{n=1}^{\infty} A_n \cos k_n x; \end{gathered} \tag{5}$$

$$y = 0 \quad T = 0; \quad y = \infty \quad T = 1, \tag{6}$$

where the A_n in (5) is the dimensionless amplitude of the n-th harmonic component of the free stream, $k_n = 2\pi/\lambda_n$ is the dimensionless wave number of the n-th component of the periodically distributed vortex (a quantity reciprocal to the dimensionless wavelength or the vortex dimension corresponding to the n-th harmonic component of the vortex), on which the conditions $k_n > 0$; $k_n = nk_1$ are imposed, where $k_1 = 2\pi/\lambda_1$ is the dimensionless wave number of the harmonic of greatest amplitude. As follows from experiments with supersonic jets impinging along the normal on a flat obstacle within the initial section [1, 2], such a fundamental harmonic is predominant in the formation of stationary flow near the obstacle. The magnitude of its wave number is of the order of $k_1 = 10^{-3}$–10^{-4}.

Let us seek the solution of (1)-(3) under the boundary conditions (5), (6) in the form:

$$\psi = x f_0(y) + \sum_{n=1}^{\infty} \frac{1}{k_n} f_n(y) \sin k_n x, \tag{7}$$

$$T = T_0(y) + \sum_{n=1}^{\infty} \theta_n(y) \cos k_n x. \tag{8}$$

Then the expressions for the velocity components and vortices are written thus:

$$\begin{gathered} v_x = x f_0' + \sum_{n=1}^{\infty} \frac{1}{k_n} f_n' \sin k_n x, \\ v_y = -f_0 - \sum_{n=1}^{\infty} f_n \cos k_n x, \end{gathered} \tag{9}$$

$$\begin{gathered} \Omega_z = x f_0'' = \sum_{n=1}^{\infty} \omega_n \sin k_n x, \\ \omega_n = \frac{1}{k_n} f_n'' - k_n f_n. \end{gathered} \tag{10}$$

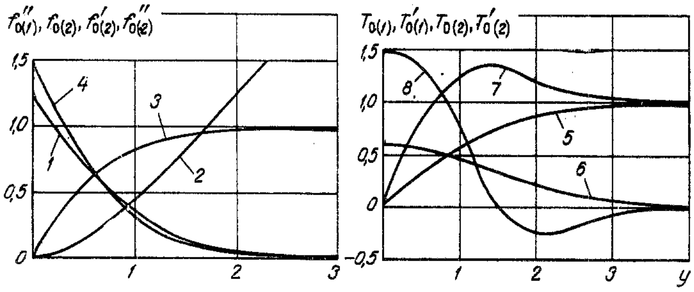

Fig. 2. First and second approximations for the zero members of the expansions (7), (8): 1) $f''_{0(1)}$; 2) $f_{0(2)}$; 3) $f'_{0(2)}$; 4) $f''_{0(2)}$; 5) $T_{0(1)}$; 6) $T'_{0(1)}$; 7) $T_{0(2)}$; 8) $T'_{0(2)}$.

Taking account of (8)-(9), we obtain the following boundary conditions for the functions from (5)-(6):

$$f_0(0) = f_0'(0) = 0; \quad f_0'(\infty) = 1; \tag{11}$$

$$f_n(0) = f_n'(0) = 0; \quad f_n'(\infty) = A_n; \tag{12}$$

$$T_0(0) = 0; \quad T_0(\infty) = 1; \tag{13}$$

$$\theta_n(0) = 0; \quad \theta_n(\infty) = 0. \tag{14}$$

Let us limit the analysis to an examination of such harmonic components of the velocity components and vortices for which $k_n \ll 1$. If (9) is substituted into (1), and terms containing k_n^2 are neglected, then the expression obtained can be integrated with respect to y. Taking account of the boundary conditions (11), (12), we obtain

$$x(f_0''' - f_0'^2 + f_0 f_0'' + 1) - \sum_n (f_0' f_n' - f_0'' f_n - A_n)\, x \cos k_n x$$
$$+ \sum_n \frac{1}{k_n} (f_n''' - f_0' f_n' + f_0 f_n'' + A_n) \sin k_n x = \sum_{n,i} \frac{1}{k_n} (f_n' f_i' - f_n'' f_i - A_i A_n) \sin k_n x \cos k_i x. \tag{15}$$

The summation over n in (15) is for such n for which the condition $k_n \ll 1$ is still conserved. If known formulas for the series expansions of the functions $\cos k_n x$ and $\sin k_n x$ in powers of $k_n x$ are utilized, then taking into account the $k_n = n k_1$, we obtain equality of the two power series in (15). Equating coefficients of identical powers of $k_1 x$, we obtain the following system of (n + 1) equations to determine f_0, f_n (n = 1, 2, 3, . . .):

$$f_0''' - f_0'^2 + f_0 f_0'' + 1 = \sum_n (2 f_0' f_n' - f_0'' f_n - f_0 f_n'' - f_n''' - 2A_n) + \sum_{n,i} (f_n' f_i' - f_n'' f_i - A_i A_n), \tag{16}$$

$$\sum_n [(f_n''' - f_0' f_n' + f_0 f_n'' + A_n) + (2j+1)(f_0'' f_n - f_0' f_n' + A_n)]\, n^{2j} = (2j+1)! \sum_{t=0}^{j} \sum_{n,i} (f_n' f_i' - f_n'' f_i - A_i A_n) \frac{n^{2t} i^{2(j-t)}}{(2t+1)!\,[2(j-t)]!}. \tag{17}$$

Analogously substituting (8), (9) into the energy equation (3), we obtain the following system of (n + 1) equations to determine T_0, Θ_n (n = 1, 2, . . .):

$$\frac{1}{\mathrm{Pr}} T_0'' + f_0 T_0' + \sum_n \left(\frac{1}{\mathrm{Pr}} \Theta_n'' + T_0' f_n + f_0 \Theta_n' \right) = -\sum_{n,i} f_n \Theta_i', \tag{18}$$

$$\sum_n \left[\frac{1}{\mathrm{Pr}} \Theta_n'' + T_0' f_n - f_0 \Theta_n' + 2j f_0' \Theta_n \right] n^{2j}$$
$$= -(2j+1)! \sum_{t=0}^{j} \sum_{n,i} \left\{ \frac{i}{n} f_n' \Theta_n \frac{n^{2t+1} i^{2(j-t)+1}}{(2t+1)!\,[2(j-t)+1]!} + f_n \Theta_i' \frac{n^{2t} i^{2(j-t)}}{(2t)!\,[2(j-t)]!} \right\}. \tag{19}$$

The system (16), (19) yields 2(n + 1) equations to determine f_0, T_0, f_n, Θ_n. The boundary conditions (11)-(14) are utilized to solve the mentioned system. In application to the case of impingement of a supersonic jet on an obstacle when the external stream contains one harmonic component of greatest amplitude, i.e., for $A_n \approx 0$, n = 2, 3, . . . , the practical convergence of the series (7)-(9) is sufficiently rapid, and as

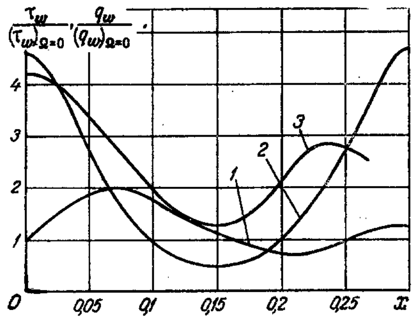

Fig. 3. Distribution of heat fluxes and friction stresses along an obstacle: 1) friction stress (computation); 2) heat flux (computation); 3) heat flux (experiment).

further computations showed, for an approximate solution it is possible to limit oneself to two terms in the mentioned series. The summation is hence over i, n = 2 in (16), (18) to determine f_0, T_0. The equations to determine f_1, f_2, Θ_1, Θ_2 are obtained from (17) and (19) for j equal successively to 1 and 2. It is assumed Pr = 1.0 in the solution.

The results of the solutions of the equations obtained under the boundary conditions (11)-(14) are presented in Figs. 1 and 2. The solution is carried out by iteration on an electronic analog computer. The first approximation for the functions is denoted by the subscript 1, and the second by the subscript 2.

In order to analyze the solution obtained, a graph was constructed of the heat flux and friction stress distribution along the obstacle in the neighborhood of the stagnation points q_w and τ_w, referred to the corresponding computed quantities obtained for the impingement of an irrotational stream on an obstacle $(q_w)_{\Omega=0}$ and $(\tau_w)_{\Omega=0}$:

$$\frac{q_w}{(q_w)_{\Omega=0}} = \left(\frac{T_0' + \sum\limits_n \Theta_n' \cos k_n x}{T_0'} \right)_{y=0},$$

$$\frac{\tau_w}{(\tau_w)_{\Omega=0}} = \left(\frac{x f_0'' + \sum\limits_n \frac{1}{k_n} f_n'' \sin k_n x}{x f_0''} \right)_{y=0}.$$

The results of the computations by the formulas presented are shown in Fig. 3 for values of the amplitude and wave number of the fundamental harmonic $A_1 = 0.5$ and $k_1 = 10^{-3}$. For comparison, results of an experiment measuring the heat fluxes on an obstacle in the neighborhood of the stagnation point of a supersonic jet are presented here [1] for M = 2.5; n = 3.0; $\eta = 2d_a$, $\varkappa = 1.4$. As is seen from the comparison, for $A_1 = 0.5$ and $k_1 = 10^{-3}$ the results of the computation are in good agreement with the results of the experiment.

The following deductions can be made on the basis of the results of the computation:

1. For a fundamental harmonic of amplitude $A_1 = 0.5$ the heat flux on an obstacle at the stagnation point, as computed by the method proposed, will exceed the corresponding quantity for an irrotational stream 4.5-fold. The increase in friction stress on the wall as compared with the corresponding quantity for an irrotational stream is not more than twofold.

2. The heat flux and friction stress vary periodically along the obstacle with approximately the same period. Alternation of the maximums and minimums of the heat flux and friction stress along the obstacle indicate the presence of periodically distributed vortices of alternating sign near the wall. The sign of A_1 indicates the direction of vortex rotation (for $A_1 > 0$ the first vortex rotates counterclockwise).

3. Agreement between the results of computing the heat flux distribution along the obstacle with the data of experiment is good enough in the domain where the first vortex is located on the obstacle. As x recedes from the first vortex, the divergence between theory and experiment increases, which is apparently associated with the more complex nature of the perturbations imposed on the external stream ($A_n \neq 0$, n = 2, 3, . . .).

NOTATION

ξ, η, ζ are the physical coordinates;
x, y, z are the dimensionless coordinates;

v_x, v_4	are the velocity components;
ψ	is the stream function;
Ω_ζ, Ω_z	are the vortex projections on ζ and z axes;
β	is the constant in the linear longitudinal stream velocity distribution in the neighborhood of the stagnation point;
T	is the temperature;
ν	is the viscosity;
Pr	is the Prandtl number;
q_w, $(q_w)_{\Omega=0}$, τ_w, $(\tau_w)_{\Omega=0}$	are the heat flux and friction, respectively, with stream vorticity taken and not taken into account;
M	is the Mach number;
n	is the jet incalculability;
η_0	is the distance between nozzle exit and obstacle;
d	is the nozzle diameter;
$\varkappa$	is the adiabatic index;
w	is the wall;
∞	is the inviscid stream;
a	is the nozzle exit.

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